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An Investigation of the Non-Convexification Effects of Filtering Techniques in Compliance Minimization Problems

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***Abstract*—Most real-life engineering optimization problems are non-convex by nature. In a topology optimization context, this non-convexity is even exacerbated by the extra restrictions imposed during the optimization process to enforce mesh-independent black/white manufacturable solutions. Such restrictions include intermediate density penalization, as well as external regulation techniques imposed to tackle some numerical instabilities such as checkerboarding and mesh dependence, in addition to various design constraints. This non-convexity gives rise to the problem of local minima, where the converged solution is greatly affected by the algorithmic parameters as well as the initial guess. To overcome this non-convexity, filtering techniques are often employed. In this work, we present a comprehensive treatment of the potential effects of non-convexification of filters on the topology optimization problem.**

***Index Terms*—topology optimization, finite element analysis, filtering techniques, numerical instabilities, convexity**

I. Introduction

Since the introduction of the **S**olid **I**sotropic **M**aterial with **P**enalization (SIMP) method in the seminal paper by Bendsøe [1], material interpolation methods have become one of the most active research areas in engineering optimization. Although the origin of almost all density-based approaches lies in linear elasticity, they have been successfully extended to even more complicated single and multiphysics fields. In a general sense, the optimal topology of a problem refers to the location and number of holes such that an objective function is extremized (c.f. Fig. 1).

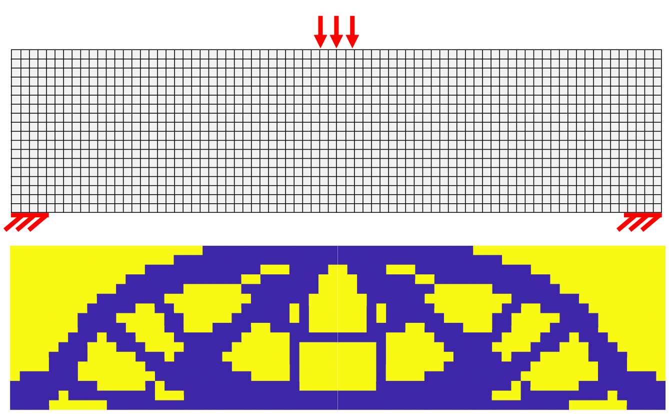

Fig. 1. Topology optimization applied to an MBB problem (a.k.a. a bridge). The top figure shows the problem description and the finite element discretization while the bottom figure shows the optimized result (yellow/blue represent void/material respectively).

To simplify the numerical implementation of topology optimization formulations, the normalized density is usually taken to be element-wise constant rather than the less popular node-wise approach. This way the design variables can be taken outside the integral of the elemental stiffness matrices (c.f. [2, p. 68] and [3, p. 1419]). It is a widely known fact that any approach that enforces discrete $0/1$ solutions is inherently non-convex. A rather unfavorable consequence of this non-convexity is the obscureness of the global optimum. In other words, the converged solution of non-global optimization approaches would be one of the huge pool of local minima the problem possesses. To complicate the problem further, global optimization approaches are extremely computationally extensive, and, so far, prove unable to handle the typically massive number of design variables in practical topology optimization problems [2, p. 74].

A major numerical instability that was early recognized as a direct consequence of the non-convexification of the objective function is the *Local Minima* problem. It mainly refers to the problem of obtaining different solutions for the same finite element (FE) discretization upon choosing different algorithmic parameters [2, p. 74]. The converged solution of an optimization problem is mainly determined by three parameters[1]; the initial guess, the optimization direction, and the optimization "speed". The latter two parameters are characteristics of the optimization algorithm/solver in use [4]. The optimization direction is determined by the gradients[2] of the objective function and constraints with respect to the

[1]Strictly speaking, this statement only applies to optimization problems with a non-changing objective function, so it doesn't strictly apply when continuation methods are used (i.e. increasing the penalization factor). The use and effect of continuation methods is discussed in depth in a later section.

[2]The discussion in this study is solely dedicated to gradient-based optimization, since the usefulness of non-gradient based methods is still unclear as was discussed in the forum article by Sigmund [5].

design variables (i.e. sensitivity analysis), while the optimization speed is determined by how aggressive the algorithm is, (e.g. how much change in the design variables is allowed per iteration).

A number of methods have been implemented to overcome the numerical instability of local minima, the most popular of which is filtering techniques. Although they have been in use in the academic and industrial communities for more than two decades, there has not been a comprehensive study of their non-convexification effects yet. In this work, we attempt to investigate the non-convexification effects of various filtering techniques and their effect of the global optimality of the topology optimization problem.

II. Mesh-Independency Filtering

Since the first appearance of mesh-independency filters (i.e. sensitivity filtering by Sigmund [6, 7]), they have become extremely popular in the research community for their effectiveness, computational efficiency, and ease of implementation. Generally, filtering methods are implemented as an intermediate step in the cycle of the analysis step (i.e. solving the equilibrium equations and calculating the original sensitivities) and the optimization step (i.e. updating the design variables). They can be divided into two categories: sensitivity filtering and density filtering[3]. Sensitivity (density) filters work through modifying each element's sensitivity (density) based on the sensitivities (densities) of neighbouring elements within a predefined filter radius. Although these filters were originally invented to prevent common numerical instabilities such as checkerboarding and mesh dependency, some filters' capabilities have been extended to implementing minimum and maximum length scales in both solid and void regions in addition to some physics-specific constraints such as preventing one-node hinges in compliant structural mechanisms. A worthy remark is that although the original filter proposed by Sigmund [6, 7] was of the sensitivity type, most filters currently in use are of the density type. One of the reasons behind this is that it's relatively easier to conceptually link the parameters used in defining the density filter to the density results, which is not easily the case with sensitivity filtering. For a comprehensive review, see [8] and references therein.

It seems prevalent that almost all the work published on black/white enforcing filters recommends using continuation methods with the filtering schemes (c.f. [9, p. 409], [10, p. 249], and [11, p. 128]). In other words, there is a consensus in the literature that sophisticated filtering schemes (i.e. filters that perform any task beyond the simple prevention of checkerboarding and mesh dependency) - if implemented without continuation - would generally cause either unstable behavior at worst, or cause convergence to local minima at best. Yet, little attention is usually given to attempting to investigate the effects of the filtering schemes on the convexity of the problem. A probable cause for this general *attitude* is the fact that the severe localized penalization imposed by density-based methods on intermediate density elements is generally much stronger and tends to overshadow any other form of non-convexification caused by additional constraints.

In the following, we attempt to investigate the effect of filtering methods on the convexity of the problem. At this point, it's worthwhile to revisit some generalizations of convex functions, namely *quasiconvex functions*. Following the treatment in Bazaraa et al. [12, p. 134], a function $f : S \to R$, where S is a nonempty convex set in R^N, is said to be *quasiconvex* if for each $\mathbf{x}_1$ and $\mathbf{x}_2 \in S$, the following inequality is true:

$$f[\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] \leq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}, \quad \text{for each } \lambda \in (0,1). \tag{1}$$

Although quasiconvex functions have a single minimum, their definition don't preclude the existence of multiple stationary points (see Fig. 2a), which is detrimental to iterative solvers. Hence, a more useful concept is the strictly quasiconvex functions. A function $f : S \to R$, where S is a nonempty convex set in R^N, is said to be *strictly quasiconvex* if for each $\mathbf{x}_1$ and $\mathbf{x}_2 \in S$ with $f(\mathbf{x}_1) \neq f(\mathbf{x}_2)$, the following inequality is true:

$$f[\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] < \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}, \quad \text{for each } \lambda \in (0,1). \tag{2}$$

By enforcing $f(\mathbf{x}_1) \neq f(\mathbf{x}_2)$ and the strictness of the inequality, we actively eliminate any stationary points except at the global minimum (see Fig. 2b). If one is interested in a generalization of convex functions that supports uniqueness of solutions, *strongly quasiconvex functions* might be of interest. A function $f : S \to R$, where S is a nonempty convex set in R^N, is said to be *strongly quasiconvex* if for each $\mathbf{x}_1$ and $\mathbf{x}_2 \in S$, with $\mathbf{x}_1 \neq \mathbf{x}_2$, the following inequality is true:

$$f[\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] < \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}, \quad \text{for each } \lambda \in (0,1). \tag{3}$$

By enforcing $\mathbf{x}_1 \neq \mathbf{x}_2$, strongly quasiconvex functions (see Fig. 2c) assert uniqueness of the global optimum. However, a major disadvantage of the above mentioned functions is that they must have convex lower level sets. That is, for S a nonempty convex set in R^N, the lower level sets defined as:

$$L_\alpha = \{x \in S \,|\, f(x) \leq \alpha\}. \tag{4}$$

for $\alpha \in R$, must be convex [13, p. 95]. This disadvantageous property is usually hard to prove for multivariate functions and doesn't add significant value for an iterative solver. Hence, it would be useful to explore a more generalized concept, that is *unimodal functions*. Bazaraa et al. [12, p. 156] define *univariate* unimodal functions as follows; a function $f : S \to R$, where S is some interval on R, is unimodal on S if there exists an $x^* \in S$ at which f attains a minimum and f is nondecreasing on the interval $\{x \in S : x^* \leq x\}$ and nonincreasing on the interval $\{x \in S : x \leq x^*\}$. As for *multivariate* unimodal functions, a number of conflicting definitions

[3] We chose to categorize Heaviside filtering as a type of density filtering since it operates on the elemental densities while some other references treat it as separate from density filtering.

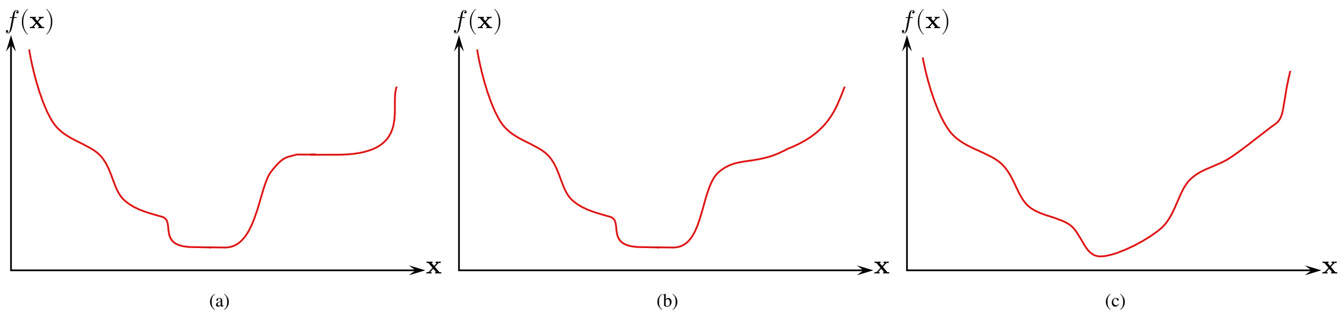

Fig. 2. Generalization of convex functions: **a.** quasiconvex, **b.** strictly quasiconvex, and **c.** strongly quasiconvex. Notice the difference between the stationary points in the three types.

exist in literature, specially in the field of Probability and Statistics [14, 15, 16]. Some authors associate the definition of unimodality with that of quasiconvexity as in having convex lower level sets, which is true for univariate functions but not for multivariate ones. In this work, for the definition of multivariate unimodal functions, the natural extension of univariate unimodal functions is utilized [16, p. 38]. That is; a multivariate function $f(\mathbf{x})$ is called unimodal if it's nondecreasing along rays emanating from its global minimum in all directions (see Fig. 3). To preclude the existence of multiple stationary points, we could utilize the notion of strict monotonicity so that $f(\mathbf{x})$ is strictly increasing in every direction emanating from its global minimum.

Returning to our discussion on the effect of filtering on convexity, it's true that convexity might be distorted. However, unimodal functions are an example of non-convex functions that still possess a global minimum attainable through iterative solvers. Hence, it's safe to assume that as long as the modified function is strictly unimodal, no new local minima would arise because of this non-convexity. It's worth noting that in penalized topology optimization, the unfiltered problem is already non-convex with many local minima. Hence, the above discussion would be only applicable to smaller *locally* convex parts of the large non-convex function (i.e. a localized valley that has a single minimum). In the following, we reflect this discussion onto sensitivity filtering.

III. Sensitivity Filtering

What the sensitivity filtering does is that it modifies the descent direction used in the optimizer, albeit without affecting the value of the objective function itself [17, p. 474]. One could say that sensitivity filtering "tricks" the optimizer by altering the gradient of the objective function. Hence, the optimizer would follow a different route towards the minimum and obviously would settle at a different minimum from the unfiltered original minimum, mainly because the sensitivity filtering flattened the objective function at this point as seen from the gradient's perspective. Since the gradient has been altered, it's safe to say that sensitivity filtering "might" cause some degree of non-convexification, albeit it cannot strictly be called penalization since the value of the objective function itself hasn't been altered.

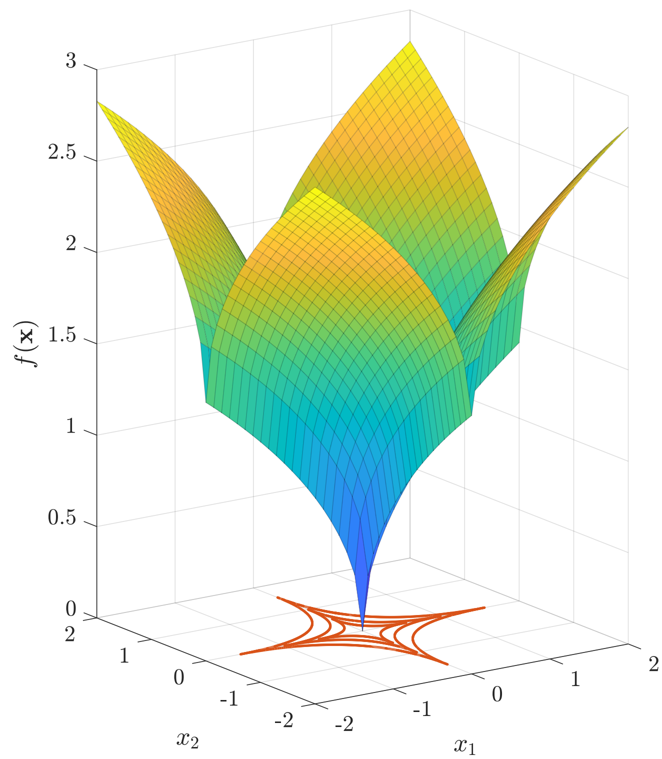

Fig. 3. Function $f(x_1, x_2) = \sqrt{|x_1|} + \sqrt{|x_2|}$ is an example of a multivariate unimodal function that doesn't belong to any category of quasiconvex functions since its lower level sets aren't convex. Yet, it has a unique global minimum that can be found using iterative solvers.

Any filtering method starts by determining the neighbourhood of each element; that is the set N_e consisting of the elements with centers spatially located within a given filter radius r of the center of element e as follows:

$$N_e = \{i \mid ||\mathbf{x}_i - \mathbf{x}_e|| \leq r\}. \tag{5}$$

where $\mathbf{x}_i$ denotes the spatial location of the center of element i.

A modified version of the sensitivity filter that accounts for non-regular meshes with varying elemental volumes is as follows [9, p. 408]:

$$\widetilde{\frac{\partial f}{\partial \rho_e}} = \frac{\sum_{i\in N_e} w_e(\mathbf{x}_i)\rho_i \frac{\partial f}{\rho_i}/v_i}{\rho_e/v_e \sum_{i\in N_e} w_e(\mathbf{x}_i)}, \tag{6a}$$

$$w_e(\mathbf{x}_i) = r - ||\mathbf{x}_i - \mathbf{x}_e||. \tag{6b}$$

where f is the objective function, ρ_e is the density of element e, $w_e(\mathbf{x}_i)$ is a linearly decaying (cone-shaped) weighting function[4] of element i within the neighbourhood of element e, v_i is the volume of element i, and $\partial f/\partial \rho_i$ and $\widetilde{\partial f/\partial \rho_i}$ denote the original and modified sensitivity respectively of the objective function with respect to element i.

A noteworthy remark is on how the descent direction is calculated inside the typical mathematical solvers used in topology optimization (e.g. OC, GCMMA, MMA, etc.). The general concept is that the solvers try to satisfy the Karush-Kuhn-Tucker conditions by utilizing a combination of the gradients of the objective function and the constraints. Subsequently, the descent direction and step are calculated based on the given move limits and the bounds of the design variables. Consequently, it is safe to say that as long as the inputted gradients denote ascent directions, the outputted updated design variables would always denote a reduction in the objective function (except at a minimum of course). Hence, our investigative approach for sensitivity filters would be focused on the input to the mathematical solvers; that is whether the filtered sensitivities still constitutes an ascent direction or not. In other words, whether the modified function is *locally* strictly unimodal or not. In mathematical terms, it means satisfying the following condition:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\rho}} \cdot \widetilde{\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\rho}}} > 0 \tag{7}$$

In order to ensure that such a dot product is always positive, we need to ensure that each multiplication term is actually positive on its own as follows:

$$\frac{\sum_{i\in N_e} w_e(\mathbf{x}_i)\rho_i \frac{\partial f}{\rho_i}/v_i}{\rho_e/v_e \sum_{i\in N_e} w_e(\mathbf{x}_i)} \cdot \frac{\partial f}{\partial \rho_e} > 0 \tag{8}$$

Since $w_e(\mathbf{x}_i)$, ρ_i, and v_i are always positive, any multiplicative combination of these quantities would always be positive. Given the fact that sensitivities are always negative in compliance minimization problems, this means that the modified sensitivities are always negative. Hence, each individual term in Eq. 8 is in fact positive as it's a multiplication of two negative quantities. This concludes the proof that the filtered sensitivities in compliance minimization problems always constitute an ascent direction (i.e. the modified function is *locally* strictly unimodal). A worthy remark is that an essential component of our proof is that in compliance minimization problems, the original sensitivities are always negative. Hence, in other problems where the sensitivities might be either positive or negative, there exists the possibility that sensitivity filtering might cause non-convexification and disturb the trajectory of the mathematical solver.

[4] The upcoming proof still applies to other weighting functions such as the Gaussian (bell-shaped) distribution and the constant weighting functions since they are always positive.

IV. Basic Density Filtering

This category includes types of density filtering that enforce a grey transition region along the boundaries. Density filtering works by mapping each design point to another design point based on the details of the density filtering scheme. This mapping results in two main effects; **(i)** a jump from the original design point to the filtered one, and **(ii)** based on this jump, the sensitivity has to be modified. A typical density filtering takes the following form [18, 19]:

$$\widetilde{\rho_e} = \frac{\sum_{i\in N_e} w_e(\mathbf{x}_i) v_i \rho_i}{\sum_{i\in N_e} w_e(\mathbf{x}_i) v_i}. \tag{9}$$

and the sensitivities are modified accordingly as follows:

$$\frac{\partial f}{\partial \rho_e} = \sum_{i\in N_e} \frac{\partial f}{\partial \widetilde{\rho_i}} \frac{\partial \widetilde{\rho_i}}{\partial \rho_e}, \tag{10a}$$

$$\frac{\partial \widetilde{\rho_i}}{\partial \rho_e} = \frac{w_e(\mathbf{x}_e) v_e}{\sum_{j\in N_i} w_e(\mathbf{x}_j) v_j}. \tag{10b}$$

Let's focus on the resulting sensitivities first, it's clear that $\partial\widetilde{\rho_i}/\partial\rho_e$ is always positive since it's a multiplicative combination of $w_e(\mathbf{x}_e)$ and v_e which are always positive. Hence, according to Eq. 7, the modified sensitivities (Eq. 10a) didn't change signs and still in fact constitute an ascent direction. This proof, unlike the sensitivity filtering proof, applies to any problem, not just to compliance minimization.

As for the modified densities, it's worthy to investigate whether the filter constitutes an affine transformation or not. A careful look at the modified densities (Eq. 9) reveals that it can be put in the form:

$$\{\widetilde{\boldsymbol{\rho}}\} = [\mathbf{A}]\{\boldsymbol{\rho}\}. \tag{11}$$

where $\mathbf{A}$ is an $N \times N$ square matrix since $\widetilde{\boldsymbol{\rho}}$ and $\boldsymbol{\rho}$ have the same dimension N. This relation constitutes a linear mapping (or an affine mapping to be more general [13, p. 79]) from the original to the modified densities. A rather nice property of affine transformations is that the resultant set is convex if the original is, i.e. the modified densities set is convex if the original densities set is convex, which it is. This concludes that the modified domain is indeed convex. As for the convexity of the codomain, since the objective function is merely evaluated at a new design point, its codomain undergoes no changes and retains its original convexity, if any existed.

Although we have proven the basic density filter maintains convexity, an intriguing question remains; does the affine mapping exclude any design points from the modified domain? In order to answer this question, we need to investigate whether the matrix $\mathbf{A}$ is invertible or not. This is mainly because a non-invertible affine mapping collapses the space along some directions, i.e. the modified set is smaller than the original one. The matrix $\mathbf{A}$ is singular if its determinant is zero, which can happen in one of two cases (or both): **(i)** a whole row/column is zero, or **(ii)** two rows/columns are linearly dependent. The first case cannot happen since the worst case scenario, i.e. having a filtering radius r that doesn't span any adjacent elements, will result in an identity matrix. Any increase in the filtering radius r would fill in more zero elements and hence it's impossible for $\mathbf{A}$ to have a whole row/column of zeros. As for the second case, the only scenario that would result in two (or more) rows/columns being linearly dependent is if two elements have the same neighbourhood of elements N_e and a constant weighting function $w_e(\mathbf{x}_i)$. The only case we can see this happening is at the free edge of a thin structure comprised of two rows of finite elements, the two elements at the edge would have the same neighbourhood of elements. Of course this is an extremely-unpractical[5] case and could be safely ignored. Hence, for all practical purposes, the modified domain of a basic density filter doesn't exclude any design points. We conjecture that with a careful choice of a weighting function combined with a specific design point, though unrealistic, could result in a checkerboard pattern as a result of filtering.

V. Filtering that Enforces 0/1 Designs

As mentioned before, sensitivity as well as basic density filters enforce a grey transition region along the boundaries. To overcome this issue, a class of filters that enforce 0/1 discrete designs were developed. The first appearance of such filters was in the pioneering work by Guest et al. [10], in which the authors used a nonlinear projection (a regularized Heaviside step function) to ensure a discrete 0/1 boundary and enforce a minimum length scale on the solid phase. A slight modification of this filter could enforce the minimum length scale to the void instead of the solid phase, c.f. Guest [11, p. 125].

It's well known that intermediate density penalization is mainly achieved by the penalization of the objective function itself, and is usually enforced gradually through a continuation method. To get rid of the grey transition regions, these filtering schemes enforce the jump to be towards a discrete rather than an intermediate density design point, which can be considered a form of penalization. Intermediate density penalization, if enforced by the filtering scheme rather than the objective function, would cause unstable behavior (c.f. Guest et al. [10, p. 249] and Sigmund [9, p. 409]). Mainly because the filtering scheme is not designed to seek the minimum discrete point of the problem and consequently its modified sensitivities wouldn't lead to that direction. Hence, it's essential that continuation is utilized in the filtering scheme so as to follow a similar (or lower) degree of penalization as that of the objective function. In what follows, a mathematical justification is presented.

Guest et al. [10, p. 248]'s Heaviside filter takes the following form:

$$\bar{\rho}_e = 1 - e^{-\beta\widetilde{\rho}_e} + \widetilde{\rho}_e\, e^{-\beta}. \tag{12}$$

where the parameter β controls the curvature of the regularization ($\beta = 0$ recovers the basic density filter and $\beta \to \infty$ recovers the Heaviside step function), and $\widetilde{\rho}$ is calculated as in Eq. 9 using a linear weighting function as in Eq. 6b (other weighting functions could also be used). The affine mapping argument used in proving the convexity of basic density filtering is not valid here since its converse is not true (i.e. not being an affine mapping doesn't necessarily mean producing a non-convex set). Hence, a different argument is needed, namely the generalization of which the affine mapping argument is a special case.

Soltan [20, p. 114] stated that "a mapping $f : R^N \to R^M$ is called *convexity-preserving* if the f-images of all convex sets in R^N are convex sets R^M". In our case, $M = N$. Hence, we can focus our attention on a single modified density $\bar{\rho}$, whether its output is a convex set or not. It's clear from the analytical form of the Heaviside filter (recovered as $\beta \to \infty$) that the result contains only two elements $\{0, 1\}$. In other words, any combination of neighbourhood elements' weights and densities would always result in a filtered density of 0 or 1. In other words, certain design points would be excluded from the modified set, and in our case the excluded points are the intermediate density ones. Hence, it's clear that the result is not a convex set, and the filter is not a convexity-preserving mapping. It's worth noting the resemblance between the effect of these filters and that of domain discretization on the problem's convexity.

VI. Other Filters

This subsection concerns the filters that don't enforce neither grey nor discrete designs. To overcome the downsides of the Heaviside filter, Sigmund [9] introduced the morphology-based operators *dilate* and *erode*, which enforce a minimum length scale on the solid and void phases respectively. These filters work as maximum (*dilate*) or minimum (*erode*) operators, meaning they will select the maximum (or minimum) value of the neighbourhood elements[6]. Later, Svanberg and Svärd [21] provided alternative definitions of the *dilate/erode* operators based on the second and third Pythagorean means (i.e. geometric and harmonic means) instead of the usual first Pythagorean mean (i.e. arithmetic mean).

[5] Even with including the elements outside boundary as void elements, the two elements could still have the same neighbourhood of elements (c.f. [9, p. 407] for more details on the treatment of mesh boundaries).

[6] Assuming constant weighting as in Sigmund [9]'s original formulation. Using a different weighting function will slightly alter this definition.

It's worth noting that these filters are nonlinear (not affine mappings), but they can produce a convex set for each $\bar{\rho}$. With various combinations of neighbourhood elements, they can produce $\bar{\rho} \in [0, 1]$. In fact, Svanberg and Svärd [21, p. 865] proved that the three dilate operators (i.e. morphology, geometric, and harmonic) are convex density filters (i.e. each $\bar{\rho}$ is a convex function of $\boldsymbol{\rho}$) through analytically proving that the Hessian is always positive semidefinite.

VII. Conclusions

Convexity is a core concept in optimization given the abundance of convex programming algorithms and the fact that global optimality can be proven easily in convex problems. However, the current formulation of topology optimization problems inherently introduces non-convexification into the problem. This non-convexification is introduced first by the simple domain discretization required to solve the problem numerically, and later by the intermediate density penalization required to enforce discrete solutions. Various design constraints can be sources of additional non-convexity. In this article, we presented a mathematical treatment on effects of some of the common filtering methods on the convexity of the problem. Aside from the filters that enforce 0/1 designs, most filtering techniques do not cause non-convexification.

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