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Development of a bi-criterion objective function for analytical inverse kinematic methods

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Abstract—Body posture predicting methods have many applications, such as the product design, ergonomic workplace design, human body simulation, virtual reality and animation industry. Analytic Inverse Kinematic (AIK) method, a convenient and time-saving method, has been widely applied to proactively estimate human body postures. It is indicated that a specific body posture can be determined by the optimization of an arbitrary objective function. The objective of this paper is the prediction of the posture of shoulder and elbow, for reaching tasks. In this paper, the joint displacement function and joint discomfort function are selected to be initially applied in the AIK method. Then, a bi-criterion objective function is proposed, by integrating the joint displacement function and joint discomfort function.

Results show that the joint displacement function does not predict accurate body postures when the torso is fixed. The joint discomfort function predicts reasonable posture, but it does not reflect the effects of the initial body posture. The accuracy of the arm postures, predicted by the proposed objective function, is the most satisfactory; while the optimal value of the coefficient, in the proposed objective function, is determined by golden section search.

Keywords-analytic inverse kinematic method; body posture optimization; objective function

I. INTRODUCTION

Proactive body posture estimation is very useful in many areas: (i) virtual design and test of products [1][2]; (ii) ergonomic workplace design [3][4]; (iii) human characters' embodiment in virtual reality platforms [5]; (iv) computer graphics, simulating human models or the models of other legged creatures [6].

The problem of determining appropriate body postures (i.e. appropriate configurations of joint-angle values), based on a desired target point position, is named as Inverse Kinematic (IK) problem [6] (as shown in figure 1). In this research, an optimization model is merged into a previous IK method, in order to increase the accuracy of the previous methods. In the optimization model, two objective functions (the joint displacement function and the joint discomfort function) are combined together, forming a bi-criterion objective function. The coefficient in the bi-criterion objective function is determined, based on experimental data of a reaching task, extracted from the publication of other researchers [7].

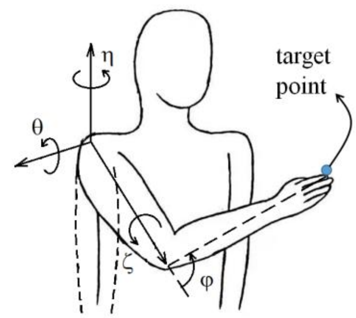

Figure 1. Inverse kinematic problem (Dashed line shows the initial position of the arm. η, θ, ζ, φ are the joint angles, which are the shoulder abduction, shoulder flexion, shoulder rotation and elbow flexion, respectively.)

II. LITERATURE REVIEW

IK is defined as a problem of solving an appropriate joint configuration based on the given position of the end effector [8]. This problem was initiated in robotics, in order to move a redundant manipulator to a desired target [6]. Apart from its use in robotics, IK has also been widely applied for human body motion in areas of computer graphics, virtual reality and ergonomic design. This paper focuses on the application of IK methods on the human body motion.

IK methods can be categorized into three major types: analytic IK methods, numerical IK methods and data-driven IK methods [5]. Analytic IK methods are meant to find out the solution as a function of the target point position. Numerical IK methods achieve satisfactory solutions through a set of iterations, while data-driven IK methods use pre-learned postures to match the positions of the end effectors. [6]

Compared with analytic IK methods, numerical IK methods can achieve better accuracy, but require 400-600 times of the time that analytic IK methods usually need [9]. When it comes to data-driven IK methods, they ensure natural body-postures [6], but need a large amount of motion data for each task, which is expensive and time-consuming to acquire

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[5]. Therefore, this research focuses on Analytic Inverse Kinematic (AIK) methods.

A. Analytic IK Method and Swivel Angle

[6]. For upper-limb applications, elbow flexion is solved first, based on the target distance from wrist joint centre to shoulder joint centre. Then, elbow joint position is limited on a circle. In order to parameterize the elbow position, the swivel angle is defined to evaluate the rotation of arm. [10][11][12]

The swivel angle is defined around the middle rotation axis (an axis pointing from shoulder joint centre to wrist joint centre), which can be determined by applying joint limits [11][12]. However, joint limits can only provide a possible range of angles, instead of a specific swivel angle [6]. Tolani et al pointed out three ways of selecting an appropriate swivel angle, from the possible range of angles: (1) select the midpoint of the possible range; (2) choose a possible value which is closest to a desired value; (3) find a value of φ which minimizes an arbitrary objective function [10]. This research focuses on the third approach, combining objective functions with the analytic IK method, in order to find an accurate solution, as well as to explore the mechanisms of the body posture determination.

B. Cost Function

Cost function is a group of functions, evaluating human performance measures (Mi, Yang, and Abdel-Malek 2009), which can be applied as the objective functions, for the determination of human body posture. This section introduces three types of commonly-applied cost functions: the delta potential energy function, joint discomfort function and joint displacement function. Their physical meanings and feasibility are introduced based on searched literature.

1) Delta potential energy

Delta potential energy describes the change of the gravity potential energy of human body, from initial posture to final posture [13]. Equation (1) shows a commonly-applied formulation of the delta potential energy function [13].

$$f_{dpe}(\boldsymbol{q}) = \sum_{i=1}^{n} (m_i \cdot g)^2 \cdot (\Delta h_i)^2 \tag{1}$$

where, m_i is the mass of the i_{th} body segment (Usually, a unit of kilogram is applied. In this research, m_i is normalized with the body mass. Therefore, body mass is applied as the unit.),

g is the gravitational acceleration (The body mass multiplied by the gravitational acceleration is body weight. Therefore, the body weight (BW) is applied as the unit of $m_i g$.),

Δh_i is the change of the height, of the centre of mass of the i_{th} body segment, from the initial body posture to the final body posture (unit: meter or millimeter; millimeter is applied in this research).

For each couple of initial target point and final target point, as the swivel angle increases, Δh_i will also increase, so that the delta potential energy of human arm will keep increasing. Therefore, the pure minimization of delta potential energy will always lead to the smallest swivel angle, which is probably not an accurate optimization.

2) Joint discomfort

The joint discomfort function has been widely applied to predict body-posture [4][13][14], which evaluates the musculoskeletal discomfort of human body [1]. Based on searched literature, the latest joint discomfort function is developed by Marler et al [15] (shown in equation (2) to (5)),

$$f_{discomf}(\boldsymbol{q}) = \frac{1}{G} \cdot \sum_{i=1}^{n} [\gamma_i \cdot (\Delta q_i^{n,norm}) + G \cdot QU_i + G \cdot QL_i] \tag{2}$$

$$\Delta q_i^{n,norm} = \frac{q_i - q_i^n}{q_i^u - q_i^L} \tag{3}$$

$$QU_i = (0.5 \cdot sin(\frac{5.0 \cdot (q_i^U - q_i)}{q_i^U - q_i^L} + 1.571) + 1)^{100} \tag{4}$$

$$QL_i = (0.5 \cdot sin(\frac{5.0 \cdot (q_i - q_i^L)}{q_i^U - q_i^L} + 1.571) + 1)^{100} \tag{5}$$

where, q_i is the value of i_{th} joint angle (unit: degree or rad),

q_i^n is the neutral value of i_{th} joint angle (unit: degree or rad),

q_i^U is the upper limit of the i_{th} joint angle (unit: degree or rad),

q_i^L is the lower limit of the i_{th} joint angle (unit: degree or rad),

$\Delta q_i^{n,norm}$ is the normalized value of the i_{th} joint angle, based on the neutral joint angle value (as shown in equation (3)). Therefore, it has no unit.

γ_i is the joint weight (without unit),

QU_i is the joint limit term expressed in equation (4),

QL_i is the joint limit term expressed in equation (5) [1].

For each joint, they evaluate its discomfort by two facts: (a) joint discomfort decreases when a segment get close to its neutral position; (b) joint discomfort rapidly increases when a segment get close to its limits [1]. Based on searched literature, its accuracy has not been validated. Therefore, the accuracy of the joint discomfort function needs to be validated.

3) Joint displacement

The joint displacement evaluates the angular displacement of each joint. In some research, the joint displacement is calculated from the neutral position [13][1], while, in other research, it is calculated from the initial position (i.e. the starting posture of the current analyzed motion or the end posture of a previous motion if a continuous motion is analyzed) [16]. When calculated from the initial position, the joint displacement represents the energy expenditure of the motion from initial posture to final posture [16], which estimates the effect of the initial posture to the final posture. Therefore, in this research, joint displacement is calculated from initial posture. Equation (6) shows a commonly-used formulation of the joint displacement function [17].

$$f_{displace}(\boldsymbol{q}) = \sum_{i=1}^{n} \omega_i \cdot \left(\Delta q_i^{i,norm}\right)^2 \quad (6)$$

$$\Delta q_i^{i,norm} = \frac{q_i - q_i^i}{q_i^u - q_i^L} \quad (7)$$

where, q_i^i is the initial value of i_{th} joint angle (unit: degree or rad),

ω_i is the joint weight of the i_{th} joint angle (with no unit).

$\Delta q_i^{i,norm}$ is the normalized value of the i_{th} joint angle, based on the initial joint angle value (as shown in (7) [1]). Therefore, it has no unit.

Zou et al determined the weights in joint displacement function by means of inverse optimization [17]. Their joint displacement function is validated in whole-body reaching tasks and predicts reasonably accurate body postures [17]. However, when the torso is fixed, its accuracy turns out to be low, which exhibits that the joint displacement function is not feasible for all types of reaching tasks. Further analysis needs to be conducted in order to determine its accuracy.

It has been reported that the combination of different cost functions (i.e. multiple objective optimization, and bi-criterion optimization [16]) is able to increase the accuracy of predicted body postures [7]. However, so far, it is still difficult to accurately determine the weights of different cost functions [18]. Therefore, in this research, selected single objective functions (the joint displacement function and joint discomfort function) were initially combined with AIK method, respectively. Then, we examined the accuracy of the body postures, predicted by the joint displacement function [17] and the joint discomfort function [15]. Then, we comprehensively combined the selected single objective functions together, proposing a new bi-criterion objective function. We have also validated the accuracy of this proposed bi-criterion objective function, and studied the effect of the weight of joint discomfort, on the accuracy of predicted joint angle.

III. METHODOLOGY

This section introduces the analysis of objective functions and how a bi-criterion objective function is developed. In order to combine the previous AIK method with an optimization model, As shown in figure 2, a Denavit-Hartenberg (DH) model [19] is established in Matlab. The parameters applied in this model are shown in table 1, where the values of L_6 and L_7 are cited from a publication of Dumas et al [20]. A shoulder joint limit model, proposed by Grassia [21], is also applied in our DH model, in order to avoid unreasonable optimizing results on shoulder rotations.

A. Simulation of previous objective functions

The feasibility of the joint displacement function and joint discomfort function is primarily judged by the experiment results of Admiraal et al [7], whose extracted data has also been applied by Kashi et al [16]. Nine subjects are involved in the experiment of Admiraal et al. Five target points are set up, thus twenty couples of initial and final target points are studied.

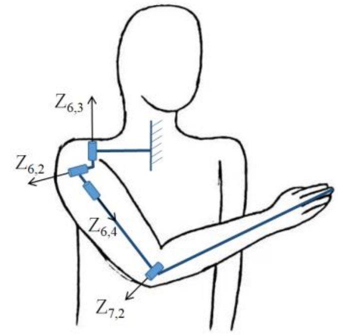


Figure 2. DH model (Adjusted index notation is applied for joint angles, in this model.)

Table 1. DH parameters

i	j	$\theta_{i,j}$	$d_{i,j}$	$\alpha_{i,j}$	$a_{i,j}$
6	3	$q_{6,3}$	0	$-\pi/2$	0
6	2	$q_{6,2}$	0	$-\pi/2$	0
6	4	$q_{6,4}$	L_6	$\pi/2$	0
7	2	$q_{7,2}$	0	0	L_7

Admiraal et al quantified human arm postures by the rotational angle of shoulder. In this simulation, measured shoulder rotation values are plotted, versus those shoulder rotation values, predicted by applying the joint discomfort function and joint displacement function, respectively. In each plot, a straight blue line, with a slope equal to one, is plotted, which indicates "measured value = predicted value".

joint weights of joint displacement function are cited from the publication of Zou et al [17], while the joint discomfort function is cited from the publication of Marler et al [15]. The delta potential energy function is not involved since it will always lead to the smallest swivel angle value.

B. Proposed bi-criterion objective function

Further simulation was conducted on the joint discomfort function, joint displacement function and delta potential energy function, for the five target points selected in the experiment of Admiraal et al [7]. Since joint discomfort curves are "well-shaped"(changes rapidly on the "wall" of these "well", but slightly on the "bottom" of these "well") (shown in section 4), a bi-criterion objective function ($f_{discomf\text{-}displace}$) has been proposed by adding joint discomfort and joint displacement together (shown in (8)), where α is a coefficient.

$$f_{discomf\text{-}displace} = \alpha \bullet f_{discomf} + f_{displace} \quad (8)$$

Then, on the "wall" of these "well", the value of this new objective function will be dependent on joint discomfort, while, on the bottom of these "well", the value of this new objective function will be mainly determined by joint displacement. Therefore, by means of this, the predicted shoulder rotation value, for each couple of initial target point and final target point, will be limited into a relatively small range, and become more accurate.

C. Search for the optimal coefficient (α_{opt}) value

In (8), if α gets close to infinite, then $f_{discomf\text{-}displace}$ will become equivalent to $f_{discomf}$; if α becomes zero, then $f_{discomf\text{-}displace}$ will become $f_{displace}$. [7]. Since the minimization of the joint discomfort function does not reflect the effect of the initial body posture, theoretically, when α keeps increasing and gets close to infinite, the accuracy of predicted shoulder rotation values will not keep increasing but start decreasing at a certain point. In other words, an optimal value of α (α_{opt}) exists between zero and infinite.

coefficient of determination (R^2) (shown in equation (9) [22], where RSS is the sum of squares of residuals, while TSS is the total sum of squares.) is calculated to quantify the accuracy of predicted shoulder rotation values. Then, R^2 should be a function of coefficient α, as shown in equation (10), whose domain is from zero to positive infinite.

$$R^2 = 1 - \frac{RSS}{TSS} \tag{9}$$

$$\boldsymbol{R^2 = R^2(\alpha)} \tag{10}$$

A pilot search is initially conducted with a step of 10000.. (i.e. A step of 4 is set for the power number of α). Then a golden section search [23] is then applied in the interval (0.0001, 10000), to find out the optimal coefficient value (α_{opt}).

A residual analysis is then conducted to compare the bi-criterion objective function and joint discomfort function. Residual ($\zeta_{residual}$) is defined as the difference between the measured values ($\zeta_{measured}$) and regressed values ($\zeta_{regressed}$) (shown in equation (11)).

$$\zeta_{residual} = \overline{\zeta}_{measured} - \overline{\zeta}_{regressed} \tag{11}$$

where, $\overline{\zeta}_{measured}$ is the mean value of measured values among the 9 subjects (unit: degree).

$\overline{\zeta}_{regressed}$ is the mean value of regressed values among the 9 subjects (unit: degree).

IV. Results and Discussion

This section shows the comparison between the shoulder rotation values, predicted by different objective functions [7]. The simulation result of different cost functions is also shown in this section.

A. Shoulder rotation values predicted by $f_{displace}$ and $f_{discomf}$

Figure 3 plots the measured shoulder rotation values, extracted from a publication of Admiraal et al [7], versus the shoulder rotation values, predicted by the joint displacement function (**Figure 3(a)**) and joint discomfort function (**Figure 3(b)**), respectively. As shown in **Figure 3(a)**, the majority of the data points spread in a triangular area, which shows that there is no obvious relation between the predicted shoulder rotation values and measured values.

This result is different from the result of Zou et al. In the research of Zou et al, the joint displacement function predicts reasonable body postures [17]. This phenomenon indicates that an IK method, validated by whole-body reaching motion, is possible to be inaccurate when the torso is fixed. When it comes to **Figure 3(b)**, the data points of all the five final target points gathered in four columns. In addition, the top of each column is approximately on the line of "measured = predicted", which shows that the accuracy of the shoulder rotation values, predicted by the joint discomfort function, turned out to be satisfactory.

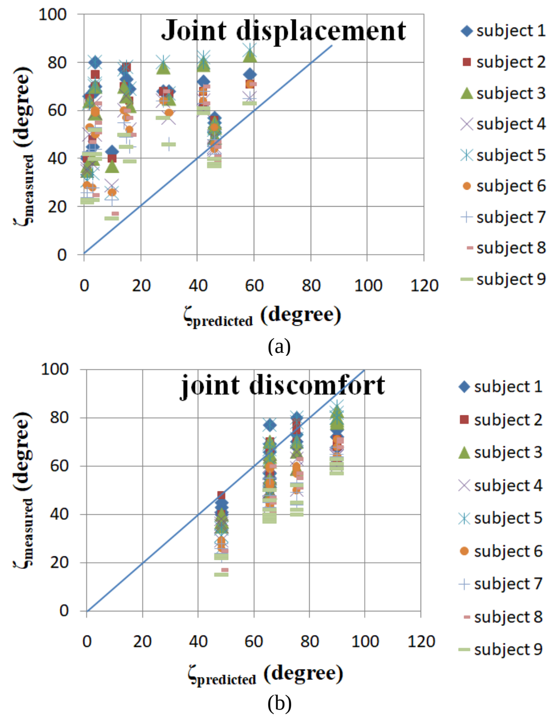


Figure 3. Comparison between predicted shoulder rotation values ((a) predicted by joint displacement; (b) predicted by joint discomfort) and the experiment data of Admiraal et al [7]

B. Simulation on different cost functions

Figure 4 shows the value of joint discomfort, joint displacement and delta potential energy, changing with the swivel angle, within joint limit, at final target 5 (The target points are cited from the publication of Admiraal et al [7].). The other four final target points give similar result. These results provide further accordance for the combination of the joint discomfort function and joint displacement function (as discussed in section 3.3). For the joint displacement function and delta potential energy function, the initial posture in this simulation is neutral standing.

C. Proposed bi-criterion objective function and optimal coefficient value

Figure 5 shows the result of the pilot search of the optimal value of α. As shown in **Figure 5,** when the value of α increased from 10^{-16} to 10^4, the R^2 value increases first, and then starts decreasing, which agrees with our hypothesis in section 3. Based on the golden section search, the optimal α

value for the subject 1, 6, 7, 8 and 9 is 13; the optimal α value for the subject 2 is 3; while the optimal value of α, for the subject 3, 4 and 5, is 1. Therefore, the global optimal value of α is determined as the weighted average, which is 7.7. **Figure 6** shows the measured shoulder rotation values, versus the predicted values, when the coefficient (α) equals to 7.7.

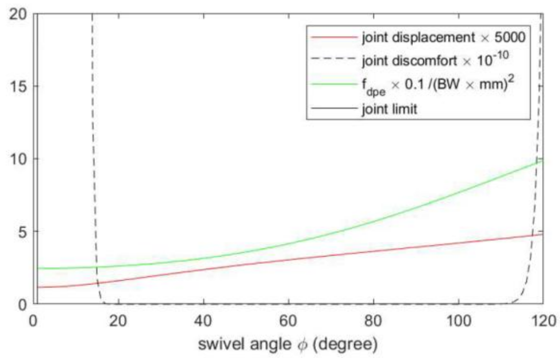


Figure 4. Change of joint displacement, joint discomfort and delta potential energy versus swivel angle, within joint limit, at final target 5

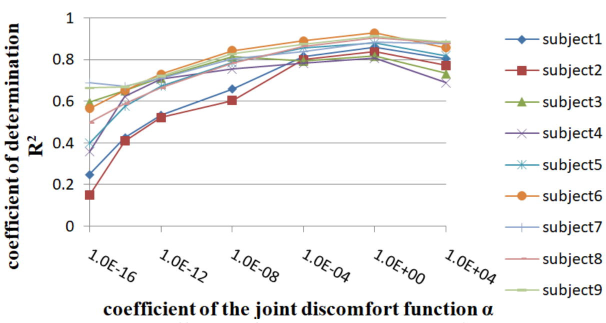


Figure 5. The coefficient of determination values of all the nine subjects, changing with the value of α.

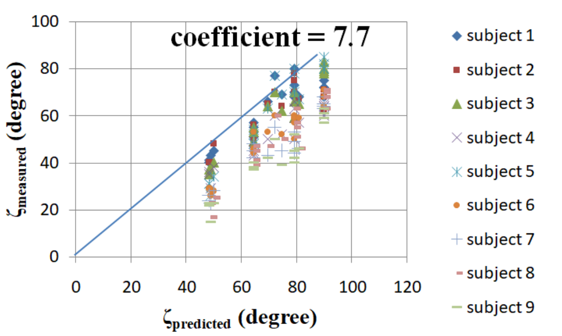
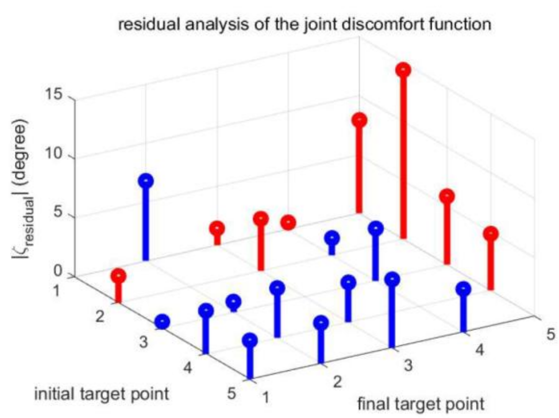


Figure 6. measured shoulder rotation values, versus the predicted values, when the coefficient (α) equals to 7.7.

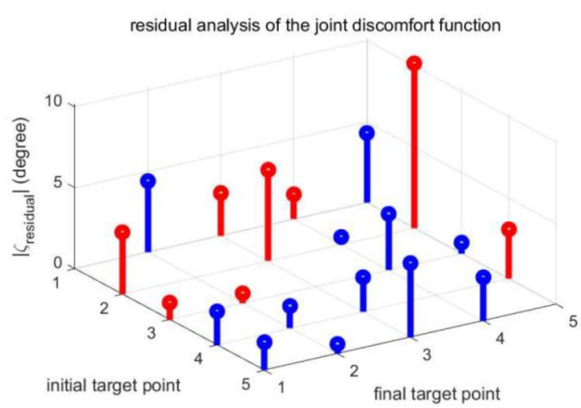
Table 2 shows the comparison between the coefficient of determination values of the shoulder rotation values, predicted by the proposed bi-criterion objective function and the joint discomfort function. It is shown that, for all the nine subjects, the proposed objective function with the optimal coefficient value predicts more accurate results. To be specific, the average coefficient of determination value, corresponding to the proposed objective function is 0.8704, increasing by 0.0573 (7%) from the value corresponding to the joint discomfort function. **Figure 7** shows the result of the residual analysis, which plots the absolute residual values ($|\zeta_{residual}|$). The red color indicates that the corresponding residual value is positive, while the blue color indicates that is negative. It is shown that, compared with the joint discomfort function, the proposed bi-criterion objective function decreases the inaccuracy of the prediction at final target 5.

Table 2. Coefficient of determination (R^2) of the shoulder rotation values, predicted by the proposed bi-criterion objective function with optimal coefficient value (α_{opt}) and the joint discomfort function ($f_{discomf}$)

Subject	R^2 ($f_{discomf}$)	R^2 (α_{opt})
1	0.8050	0.8636
2	0.7719	0.8375
3	0.7326	0.8144
4	0.6888	0.8011
5	0.8200	0.8793
6	0.8573	0.9292
7	0.8777	0.8866
8	0.8798	0.9070
9	0.8847	0.9146
Mean value	0.8131	0.8704



(a)



(b)

Figure 7. residual analysis. (a) residual analysis for the joint discomfort function; (b) residual analysis for the bi-criterion objective function

V. SUMMARY AND CONCLUSIONS

In this research, a DH model is established in Matlab, with a shoulder joint limit. Based on the established model, different objective functions are applied in AIK methods.

The joint displacement function and joint discomfort function are examined at first. The joint displacement function does not predict accurate body postures while the joint discomfort function predicts reasonable posture. However, the joint discomfort function does not reflect the effects of the initial body posture, on the final body posture.

A bi-criterion objective function is proposed by adding the joint discomfort function and joint displacement function. How the joint discomfort function limits the predicted shoulder rotation values into more accurate range has been explained, which makes our bi-criterion objective function mathematically comprehensive. Results show that the accuracy of the proposed objective function is satisfactory. We have also applied golden section search to determine the value of the coefficient, in the proposed objective function.

In this paper, we only focus on the reaching task conducted by fingertips, when the torso fixed. Therefore, the future research will include more joints. The program can also be further improved, in order to automatically search for the optimal coefficient value.

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