Proceedings of the Canadian Society for Mechanical Engineering International Congress 2020
CSME Congress 2020
June 21-24, 2020, Charlottetown, PE, Canada

Robustness analysis of a set of Second Order Sliding Modes: numerical experiments

Raúl Santiesteban-Cos
Department of Mechanical Engineering
University of New Brunswick
Fredericton, NB, Canada
rsanties@unb.ca

J. A. Carretero
Department of Mechanical Engineering
University of New Brunswick
Fredericton, NB, Canada
juan.carretero@unb.ca

***Abstract*—In this paper, a set of second order sliding mode controllers affected by disturbances with linear and nonlinear growth is under study. Mechanical systems described using a double integrator affected by external perturbations are considered in this work. A preliminary study shows that, using a nonsmooth strict Lyapunov function, it is possible to ensure finite-time stability and estimate the convergence time of the closed loop system in spite of disturbances that grow linearly and nonlinearity with respect to the state. The performance and robustness properties of the feedback synthesis are illustrated with numerical experiments solving the tracking problem for a one-link pendulum as a test system. Three experiments are carried out: the nominal system, the system affected by nonvanishing perturbations and the system affected by linearly and nonlinearity growing perturbations with respect to the state.**

***Index Terms*—Second-order sliding modes; Lyapunov function; numerical analysis**

I. Introduction

The implementation of a feedback synthesis on a mechanical system or any process is always affected by external perturbations such as friction, growing uncertainties or physical constraints. Single order sliding mode (SOSM) controllers are a good alternative because of their properties such as finite-time convergence to the origin in spite of bounded, nonvanishing external disturbances and parametric uncertainties. Moreover, a smooth version of the well known "twisting" controller has been under study considering these properties (see [2], [3], [8], [9], [11] and references quoted therein). In this paper, a generalization of the twisting algorithm plus a proportional-derivative (PD) controller is studied in order to design a robust, finite-time stabilizing set of feedback controllers.

In [12], nonsmooth strict Lyapunov functions were presented in order to assess the stability of the well known "twisting algorithm" plus a PD term. It was possible to show finite-time stability of the closed loop system in spite of perturbations that linearly grow with respect to the state. In [13], a nonsmooth strict Lyapunov function was identified to show global finite-time stability of the closed loop system using a family of controllers, in spite of nonlinear growing perturbations. Moreover, an estimation of the convergence time of the trajectories of the closed loop systems is obtained. Moreover, in [13] some particular cases of the same family of controllers where under study to observe its effect on the reaching time, the chattering amplitude, and the stationary state error. On the other hand, there have been no analyses for nonvanishing perturbations or perturbations that grow linearly with respect to the state for this set of controllers. In [1], a nonsmooth weak Lyapunov function was presented to show global asymptotic stability of the closed loop system using a set of controllers plus a PD algorithm, in spite of nonlinear growing perturbations.

Based on this, the main contribution of this paper is two fold: (a) to provide a sketch of the stability assessment of the closed-loop system, and (b) to illustrate, using numerical experiments, the performance of the control design of the disturbed and disturbance-free scenarios. A continuous stabilizing feedback controller is implemented on a one-link pendulum, affected by Coulomb friction. Three experiments are under study: nominal system, system affected by nonvanishing perturbations, and a system affected by linearly and nonlinearly growing perturbations with respect to the state.

The structure of this paper is as follows: basic assumptions of the systems under interest and some mathematical background are given in Section II. In Section III, a sketch of the stability of the unperturbed and the system affected by bounded perturbations are analysed. In both cases, finite-time stability for the origin of the closed loop system is concluded. In order to support the theoretical results, the proposed control law is implemented in Section IV, on a one-link pendulum, affected by external perturbations, as a testbed. Finally, Section V presents the conclusions of this work.

II. Preliminaries

In what follows, the definitions and mathematical background, as well as the class of systems considered throughout the rest of the paper are briefly introduced.

A. Considered systems

The general model of a mechanical system of one degree of freedom will be considered. It is described by equations of the form

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= f(x,y) + g(x,y)\tau + \delta(x,y,t),\end{aligned} \tag{1}$$

with the smooth function $g(x,y) \neq 0 \; \forall x, y$ in its domain. The origin of the unforced/undisturbed system (1) is an equilibrium point (*i.e.*, $f(0,0)=0$) and $x, y \in I\!R$ are scalar state variables. The function $f(x,y)$ represents the nominal known part of the system dynamics, which can be discontinuous, and $\delta(x,y,t)$ represents the uncertainties such as growing perturbations with respect to the state variables (see [10]).

Since the right-hand side of the Equation (1) has discontinuous terms, their solutions are understood in the Filippov sense (see [7]). It is assumed that the full vector state of the dynamic system (1) is available for measurement; note that this assumption is not restrictive because there are several works in the literature about observers and differentiators design (see [4], [5], [6], [10], [14], and references quoted therein).

For system (1) the following controller design is proposed

$$\tau = \frac{1}{g(x,y)}\left(U - f(x,y)\right), \tag{2}$$

where $U \in I\!R$ is a new control input given by

$$U = -k_1|x|^{\frac{\alpha}{2-\alpha}}\text{sgn}(x) - k_2|y|^{\alpha}\text{sgn}(y) + \rho x + \sigma y, \tag{3}$$

with k_1, k_2, ρ, and σ as positive constants and $0 \le \alpha < 1$.

Let us consider an external bounded perturbation $\delta(x,y,t)$ given by

$$|\delta(x,y,t)| \le \mu_x|x|^{\frac{\alpha}{2-\alpha}} + \mu_y|y|^{\alpha} + \mu_\rho x + \mu_\sigma y, \tag{4}$$

where $\mu_y, \mu_x, \mu_\rho, \mu_\sigma \in I\!R^+$ are positive constants. Therefore, system (1) can be reduced to the disturbed double integrator:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -k_1|x|^{\frac{\alpha}{2-\alpha}}\text{sgn}(x) - k_2|y|^{\alpha}\text{sgn}(y) \\ &- \rho x - \sigma y + \delta(x,y,t). \end{aligned} \tag{5}$$

The mathematical background and previous results for the considered systems will be state in the next section.

B. Mathematical Background

Some contributions, fundamental for the rest of this work, are now recalled. The notation of some theorems was modified for readability.

The disturbed double integrator, described by equations of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -k_1|x|^{\frac{\alpha}{2-\alpha}}\text{sgn}(x) - k_2|y|^{\alpha}\text{sgn}(y) + \delta(x,y,t), \end{aligned} \tag{6}$$

was under study in [13].

Theorem 1: [13] Let

$$\begin{aligned} V(x,y) &= \frac{2-\alpha}{2}(k_1 - \mu_x\text{sgn}(x))^2|x|^{\frac{4}{2-\alpha}} \\ &+ (k_1 - \mu_x\text{sgn}(x))|x|^{\frac{2}{2-\alpha}}y^2 + |x|^{\frac{3}{2-\alpha}}\text{sgn}(x)y \\ &+ \frac{1}{2(2-\alpha)}y^4 \end{aligned} \tag{7}$$

System (6) has finite-time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \le \frac{1}{\zeta_{minp}}\gamma_{max}^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(x(0),y(0)), \tag{8}$$

as an estimation of the convergence time of the trajectories of the closed loop system (5), with

$$\begin{aligned} \zeta_{minp} &= \min\left\{\eta_x\eta_y - \frac{3}{2}\frac{1+\alpha}{2-\alpha}, \frac{1}{2-\alpha}\left(\eta_y - \frac{3}{2}(1-\alpha)\right),\right. \\ &\quad \left.\eta_x - \eta_y\frac{1}{1+\alpha},\; \eta_x - \frac{\alpha}{1+\alpha}\right\}, \\ \gamma_{max} &= \max\left\{\lambda_{max}(P_2), \frac{1}{2(2-\alpha)}\right\}, \end{aligned}$$

and

$$P_2 = \begin{pmatrix} \frac{2-\alpha}{2}(k_1 - \mu_x\text{sgn}(x))^2 & \frac{1}{2} \\ \frac{1}{2} & \frac{2-\alpha}{2}(k_1 - \mu_x\text{sgn}(x)) \end{pmatrix},$$

$$\begin{aligned} k_1 &> \mu_x + \frac{1}{1+\alpha}\max\left\{\eta_y,\; \alpha,\; \frac{1+\alpha}{(2-\alpha)^{\frac{2}{3}}}\right\}, \\ k_2 &> \mu_y + \frac{3}{2}(1-\alpha), \\ \eta_x\eta_y &> \frac{3}{2}\left(\frac{1+\alpha}{2-\alpha}\right), \eta_x = k_1 - \mu_x,\; \eta_y = k_2 - \mu_y, \end{aligned}$$

for $0 \le \alpha < 1$ and asymptotic stability for $\alpha = 1$.

In this work, numerical results of a generalization of the set of control laws in [13] are obtained using a one-link pendulum as a testbed. A PD control law is added to the set of controllers (6) in order to compensate external disturbances described by Equation (4). A sketch of the main results of this paper will be developed in the following sections.

III. Main results

A. The Unperturbed System

Consider the unperturbed system (5) given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -k_1|x|^{\frac{\alpha}{2-\alpha}}\text{sgn}(x) - k_2|y|^{\alpha}\text{sgn}(y) + \rho x + \sigma y, \end{aligned} \tag{9}$$

where k_1, k_2, rho, σ are positive constants and $0 \le \alpha < 1$.

Theorem 2: Let

$$\begin{aligned} V(x,y) =& \frac{2-\alpha}{2}k_1^2|x|^{\frac{4}{2-\alpha}} + k_1|x|^{\frac{2}{2-\alpha}}y^2 + |x|^{\frac{3}{2-\alpha}} \\ &+ \frac{1}{2(2-\alpha)}y^4 + \left(\frac{2-\alpha}{3-\alpha} + \text{sgn}(x)\right)k_1\rho|x|^{\frac{2(3-\alpha)}{2-\alpha}} \\ &+ \frac{2-\alpha}{5-\alpha}\sigma|x|^{\frac{5-\alpha}{2-\alpha}} + \frac{1}{2-\alpha}\rho x^2y^2 + \frac{1}{2(2-\alpha)}\rho^2x^4 \end{aligned} \tag{10}$$

a strict nonsmooth Lyapunov function for the system (9) , with $\gamma > 0$. Then, the system (9) has finite-time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \le \gamma V^{\frac{1-\alpha}{4}}(x(0),y(0)), \tag{11}$$

as an estimation of the convergence time, with γ as a function of the controller gains and bounded disturbances.

For readability, the sketch of this theorem is not included. However, in the next section, a sketch of the proof and a stability assessment of the perturbed system will be treated considering external bounded perturbation that grows nonlinearly and linearly with respect to the state.

B. The Perturbed System

In this section a new strict nonsmooth Lyapunov function is proposed to show finite-time convergence of the trajectories of the perturbed system (5) to the point $(x, y) = (0, 0)$ in spite of external perturbations bounded by inequality (4).

Theorem 3: Let

$$\begin{aligned} V(x,y) = &\frac{2-\alpha}{2}(k_1 - \mu_x \mathrm{sgn}(x))^2 |x|^{\frac{4}{2-\alpha}} \\ &+(k_1 - \mu_x \mathrm{sgn}(x))|x|^{\frac{2}{2-\alpha}} y^2 + |x|^{\frac{3}{2-\alpha}} \mathrm{sgn}(x) y \\ &+ \left(\frac{2-\alpha}{3-\alpha} + \mathrm{sgn}(x)\right)(k_1 - \mu_x \mathrm{sgn}(x))(\rho - \mu_\rho)|x|^{\frac{2(3-\alpha)}{2-\alpha}} \\ &+ \frac{2-\alpha}{5-\alpha}(\sigma - \mu_\sigma)|x|^{\frac{5-\alpha}{2-\alpha}} + \frac{1}{2-\alpha}(\rho - \mu_\rho)x^2 y^2 \\ &+ \frac{1}{2(2-\alpha)}(\rho - \mu_\rho)^2 x^4 + \frac{1}{2(2-\alpha)} y^4 \end{aligned} \tag{12}$$

a strict nonsmooth Lyapunov function of the system (5). Then, the system (5) has finite-time convergence to the point $(x, y) = (0, 0)$ with

$$t_{reach} \leq \gamma_p V^{\frac{1-\alpha}{4}}(x(0), y(0)), \tag{13}$$

as an estimation of the convergence time of the trajectories of the closed loop system (5), with γ_p as a function of the controller gains and bounded disturbances.

Sketch of the proof: The proof is very similar to the disturbance-free scenario. The disturbed nonsmooth dynamics (5) is analyzed using a nonsmooth strict Lyapunov function (12). Finite-time convergence of the trajectories of the closed loop system (5) is shown by demonstrating that the time derivative of the Lyapunov function (12) is negative definite. It is possible to rewrite the time derivative of the nonsmooth Lyapunov function in terms of the same Lyapunov function, and by means of a comparison system, obtain an estimation of the time convergence of the trajectories to the point $(x, y) = (0, 0)$ in spite of the bounded perturbations (4).

In the next section, in order to support theoretical results, a numerical experiment is under study using an one link pendulum system as testbed.

IV. Numerical Experiments

The tracking problem of the one-link pendulum system, affected by Coulomb friction and external perturbations is considered to illustrate the controller performance. Figure 1 shows the considered test equipment, a simple pendulum, modelled by

$$(ml^2 + J)\ddot{q} = mgl\sin(q) - F(\dot{q}) + \tau + \delta(t, q, \dot{q}). \tag{14}$$

The friction force F is described by

$$F(\dot{q}) = \rho_v \dot{q} + \rho_c sign(\dot{q}), \tag{15}$$

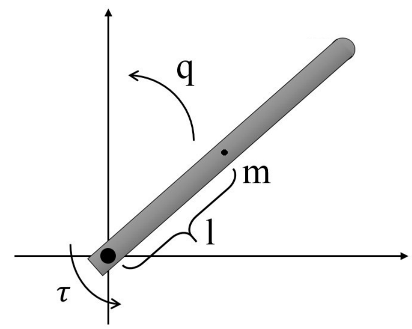

Fig. 1: Pendulum system.

where q is the angle of the pendulum with respect to the vertical, g is the gravity acceleration and τ is the control torque. The control objective is to drive the one-link pendulum to a known trajectory $r(t)$ in exact finite-time, i.e., to find a suitable control input $\tau(t)$ of the form (2)-(3) for system (1) such that the trajectory tracking error

$$e(t) = x(t) - r(t), \tag{16}$$

satisfies

$$lim_{t\to\infty} e(t) = 0,$$

even in the presence of an admissible external disturbance $\delta(x, y, t)$ of the form of equation (4).

The state-space representation of (14) is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{1}{(ml^2 + J)}\Big(mglsin(x) - \rho_v y - \rho_c sign(y) \\ &\quad + \tau + \delta(t, x, y)\Big). \end{aligned} \tag{17}$$

Lets consider the following reference trajectory for the experiments

$$r(t) = 0.25\sin\left(2\pi t + \frac{\pi}{4}\right) \tag{18}$$

even in the presence of an admissible external disturbance (4). The desired trajectory allows to have an initial deviation of $\frac{\pi}{4}$ rad at the initial time t_0.

The error dynamics (16), using equations (17) and (18) is described by

$$\begin{aligned} (ml^2 + J)\ddot{e} = &\, mglsin(x) - \rho_v y - \rho_c sign(y) \\ &+ \tau + \delta(x, y, t) - (ml^2 + J)\ddot{r}. \end{aligned} \tag{19}$$

The control law is set in the form of equations (2)-(3) as

$$\begin{aligned} \tau &= \left(ml^2 + J\right)\ddot{r} - mglsin(x) + \rho_v y + U, \\ U &= -k_1 |e|^{\frac{\alpha}{2-\alpha}} sign(e) - k_2 |\dot{e}|^{\alpha} sign(\dot{e}) - \rho e - \sigma \dot{e}, \end{aligned} \tag{20}$$

with $\alpha = 0.5$.

In this work, three experiments were considered to show the performance of the closed-loop scheme:

N_1) A nominal system. The pendulum was controlled to track a given sinusoidal trajectory without disturbances (see Figures 2 through 5).

N_2) A disturbed system affected by a perturbation bounded by a positive constant, *i.e.*, $\delta(x, y, t) = 2$ at $t = 10s$ (see Figures 6 through 9).

N_3) A disturbed system affected by disturbances that grow linearly and nonlinearly with respect to the state as in inequality (4) (see Figures 10 through 13) with the following parameters.

The initial conditions for the pendulum were fixed as $x(0) = 0$ rad and $y(0) = 0$ rad/sec. The controller gains are set to the values $k_1 = 3$, $k_2 = 1$, $\mu_x = 1$, $\mu_y = 1$. The numerical experiments below were performed using the following system values: $m = 1$, $k = 1$, $l = 1$, $g = 9.81$, $J = 1$. The simulations were done in Matlab/Simulink, using a Runge-Kutta solving method, with $0.0001ms$ time step.

First, an experiment N_1 with a set of controllers, with $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9\}$ considering no disturbance is under study, *i.e.*, $\delta(x, y, t) = 0$. The position and velocity are shown in Figures 2 and 3. Figure 4 shows the tracking error of the closed loop system, and Figure 5 shows the control input. It is possible to see that changing the value of α the robustness of the system also changes. A value of α closer to zero, the controller behaves as an ideal sliding mode controller. A value of α closer to one, a PD-like performance is obtained.

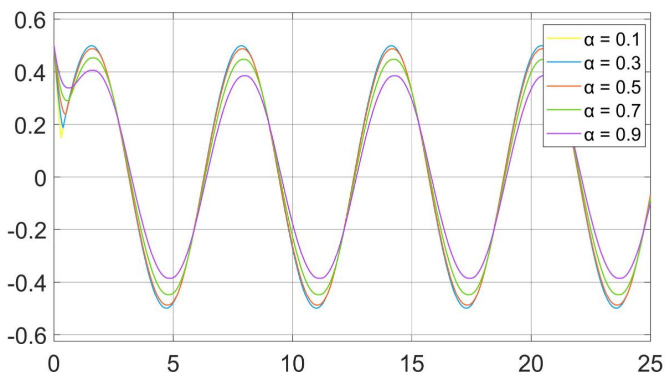

Fig. 2: Time history of the angular position in rad.

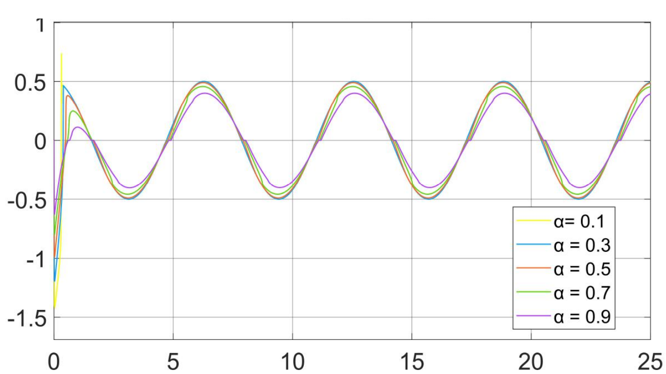

Fig. 3: Time history of the angular velocity in rad/s.

In the second set of the experiments, an external perturbation N_2, bounded by a positive constant, is added, *i.e.*,

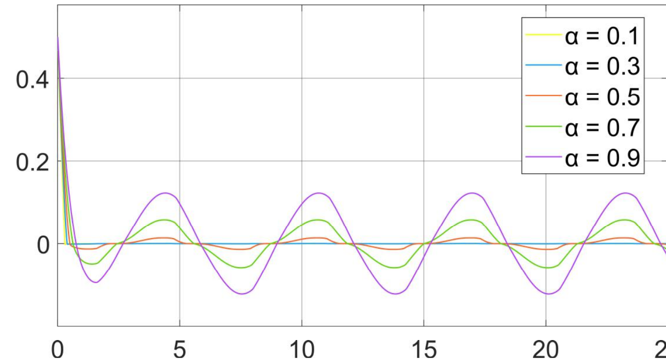

Fig. 4: Time history of the position error in rad.

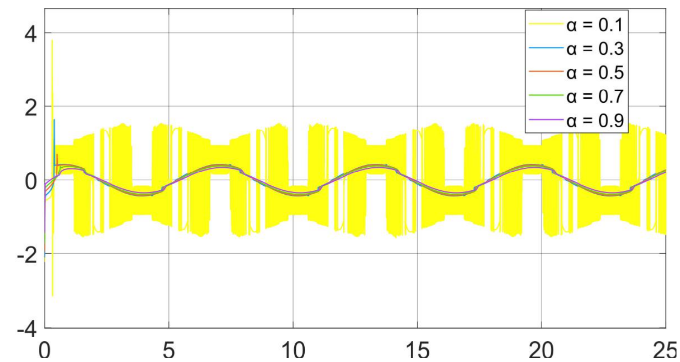

Fig. 5: Time history of the control input in Nm.

$\delta(x, y, t) = 2$ at $t = 10s$. The position and velocity are shown in Figures 6 and 7. Figure 8 shows the tracking error of the closed loop system, and Figure 9 shows the control input. A similar conclusion is visualized: Setting α closer to zero, the controller behaves more like a sliding mode controller, then the controller is robust against nonvanishing perturbations with known upper bound. A value of α closer to one, a PD-like performance is obtained, therefore the controller loose robustness against this kind of perturbations. This performance was expected since the robustness of sliding mode and PD controllers are well known.

Figures 10 to 13 depict the numerical results of the closed-loop pendulum system for scenario (S3). In this set of experiments, an external perturbation bounded by nonlinear and linear functions with respect to the state are added as Equation (4), *i.e.*, $\delta(x, y, t) = 2|x|^{\frac{\alpha}{2-\alpha}} + 2|y|^{\alpha} + 2x + 2y$ at $t = 10s$. The position and velocity are shown in Figures 10 and 11. Figure 12 shows the tracking error of the closed loop system, and Figure 13 shows the control input. The conclusion of these numerical experiments is as follows: Setting α closer to zero, the controller looses robustness against perturbations with linear/nonlinear growth with respect to the state. A value of α closer to one, PD-like performance, the controller does not loose robustness against these perturbations. This effect was also expected: for sliding mode controller is hard to fix the gain controller when the perturbations do not have a constant as an upper bound. However, the PD-like controllers can be robust against this kind of perturbations.

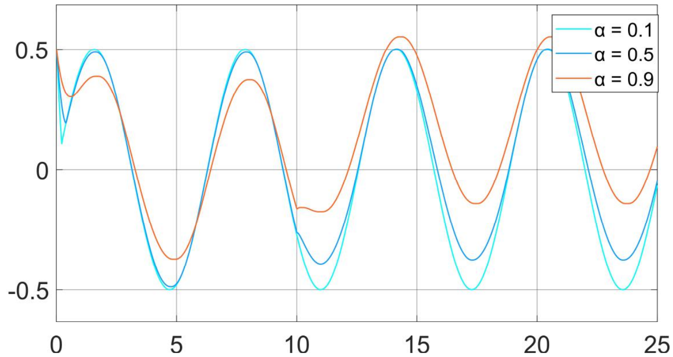

Fig. 6: Time history of the angular position in rad.

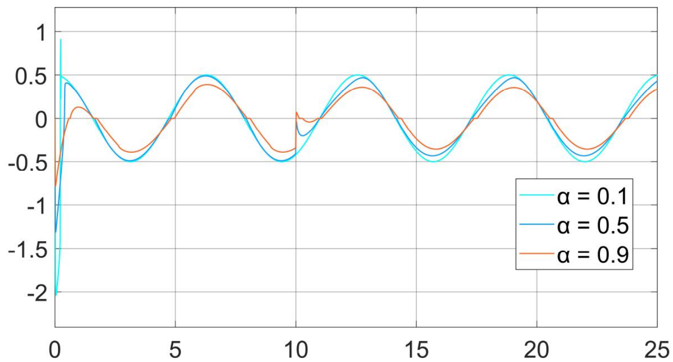

Fig. 7: Time history of the angular velocity in rad/s.

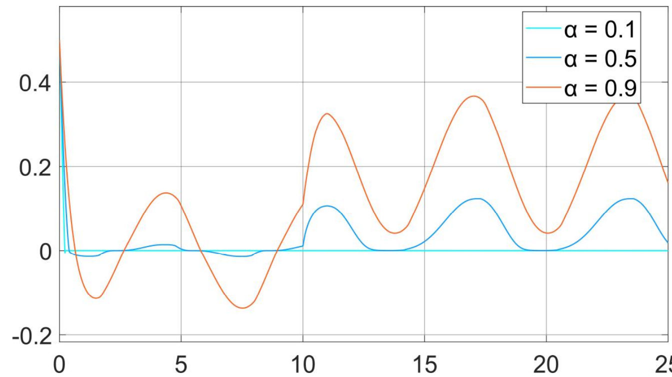

Fig. 8: Time history of the position error in rad.

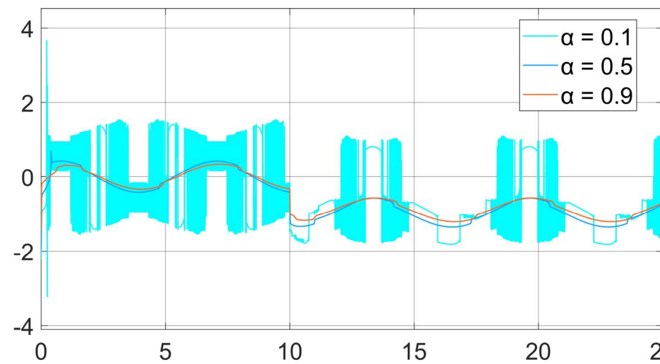

Fig. 9: Time history of the control input in Nm.

V. CONCLUSIONS

A generalization of a second-order sliding mode controller "Twisting" plus a PD algorithm is presented. A draft of two nonsmooth strict Lyapunov functions is shown in order to assess global finite-time convergence in spite of nonlinearly and linearly growing perturbations. The proposed nonsmooth strict Lyapunov functions allow an estimation of the upper bound of the convergence time. A robustness analysis of the proposed algorithm was shown through numerical experiments. The tracking control problem of a one-link pendulum in spite of perturbations with linear/nonlinear growth with respect to the state and parametric perturbations was considered as a testbed. The closed loop mechanical system showed to be robust and provide nice performance in spite of perturbations that grow with respect to the state, which is not the case for the closed loop system affected by nonvanishing perturbations. For future work, this result can be easily generalized for multidimensional case. Moreover, it can be extended when a state variable is not available for measurement, then an observer could be implemented.

REFERENCES

[1] M. Beltrán-Cardenas. Síntesis discontinua de un sistema mecánico con impactos: Bípedo de tres eslabones. Master's thesis, Tecnológico Nacional de México: Campus Culiacán, 2017.

[2] S. Bhat and D. Bernstein. Continuous finite-time stabilization of the translational and rotationnal double integrator. *IEEE Transactions on Automatic Control*, 43:678–682, 1998.

[3] I. Boiko and L. Fridman. Analysis of chattering in continuous sliding-mode controllers. *IEEE Transactions on Autonatic Control*, 50(9):1442–1446, 2005.

[4] W. Cai, J. She, M. Wu, and Y. Ohyama. Disturbance suppression for quadrotor UAV using sliding-mode-observer-based equivalent-input-disturbance approach. *ISA Transactions*, 2019.

[5] J. Davila, L. Fridman, and A. Levant. Second-order sliding-mode observer for mechanical systems. *IEEE Transactions on Automatic Control*, 50:1785–1789, 2005.

[6] C. Edwards and Y. Shtessel. Adaptive continuous higher order sliding mode control. *19th World Congress The International Federation of Automatic Control*, pages 10826–10831, 2014.

[7] A. F. Filippov. *Differential equations with discontinuous right-hand sides*. Kluwer Academic Publisher, Dordrecht, 1988.

[8] I. Furtat, Y. Orlov, and A. Fradkov. Finite-time sliding mode stabilization using dirty differentiation and disturbance compensation. *International Journal of Robust and Nonlinear Control*, 29(3):793–809, 2019.

[9] Y. Orlov. Finite-time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Control and Optimization*, 43:1253–1271, 2005.

[10] Y. Orlov, Y. Aoustin, and C. Chevallereau. Finite time stabilization of a perturbed double integrator—part i: Continuous sliding mode-based output feedback synthesis. *IEEE Transactions on Automatic Control*, 56(3):614–618, 2011.

[11] A. Polyakov, Y. Orlov, H. Oza, and S. Spurgeon. Robust finite-time stabilization and observation of a planar system revisited. In *54th IEEE Conference on Decision and Control (CDC)*, pages 5689–5694. IEEE, 2015.

[12] R. Santiesteban, L. L. Fridman, and J. A. Moreno. Finite time convergence analysis for twisting controller via a strict lyapunov function. In *Proceedings of 11th Workshop on Variable Structure Systems*, Mexico City, 2010.

[13] R. Santiesteban-Cos, A. Gárate-García, and O. Montano-Godinez. Stability analysis of a family of continuous state feedback synthesis: Theory and experiments. *International Journal of Control, Automation and Systems*, 15(3):1011–1019, 2017.

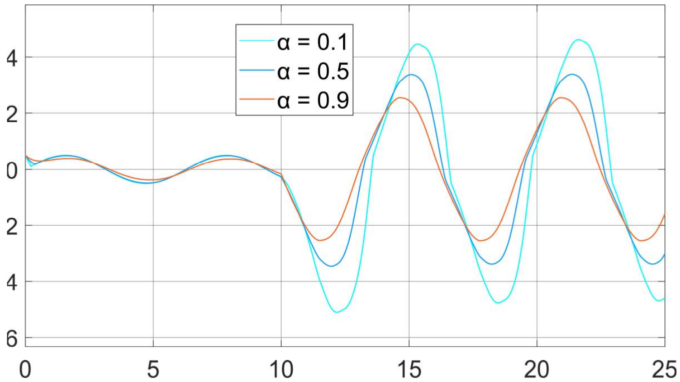

Fig. 10: Time history of the angular position rad.

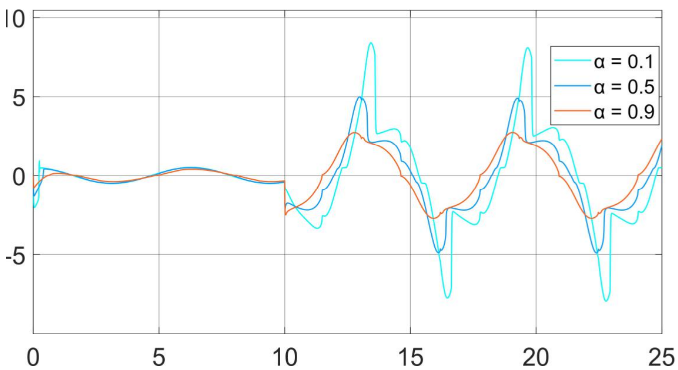

Fig. 11: Time history of the angular velocity rad/s.

[14] L. Xin and Z. Bin. Sensorless adaptive sliding mode fcs–mpc using extended state observer for pmsm systerm. In *IEEE International Conference of Intelligent Robotic and Control Engineering (IRCE)*, pages 171–177. IEEE, 2018.

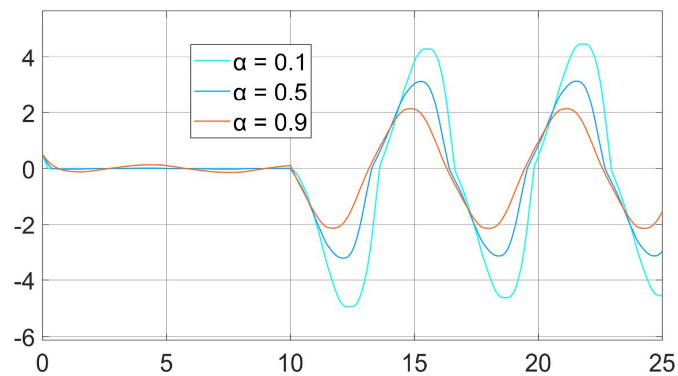

Fig. 12: Time history of the position error in rad.

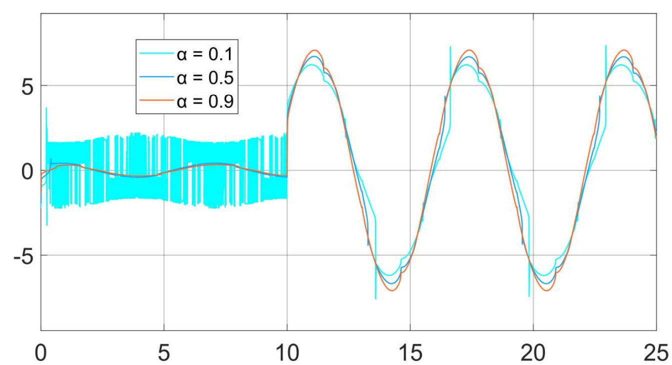

Fig. 13: Time history of the control input Nm.